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Moduli and Modularity of $(\mathbf{Q}, F)$-abelian varieties of GL_2-type

中央大学理工　百瀬 文之　(Fumiyuki Momose)
早稲田大学理工 志村 真帆呂 (Mahoro Shimura)

1 Introduction

Definition **($\mathbf{GL_2}$-type)** $A/\mathbf{Q}$: *$\mathbf{Q}$-simple abelian variety of* GL_2*-type,*

(1) $K = \mathrm{End}^0_{\mathbf{Q}}(A) := (\mathrm{End}_{\mathbf{Q}}(A)) \otimes_{\mathbf{Z}} \mathbf{Q}$: *a number field,*
(2) $dim A = [K : \mathbf{Q}]$.

$T_l(A)$: Tate-module of A, $V_l(A) := T_l(A) \otimes_{\mathbf{Z}_l} \mathbf{Q}_l$.

Remark 1 *If $A/\mathbf{Q}$ is* GL_2*-type, it has the following representation.*
$\rho_l : G_{\mathbf{Q}} := \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{Aut}_K V_l(A) = \prod_{\lambda | l} \mathrm{Aut}_{K_\lambda} V_\lambda(A)$.
$V_\lambda(A) := V_l(A) \underset{K \otimes \mathbf{Q}_l}{\otimes} K_\lambda$, $\mathrm{Aut}_{K_\lambda} V_\lambda(A) \cong \mathrm{GL}_2(K_\lambda)$.

1.1 Classification of GL_2-type

(1) $A/\mathbf{Q}$: non-CM
$A \underset{/\overline{\mathbf{Q}}}{\sim} B^r$, where '$\underset{/\overline{\mathbf{Q}}}{\sim}$' means isogenous over $\overline{\mathbf{Q}}$.
$F := Z(\mathrm{End}^0(A))$: totally real field.

$$\mathrm{End}^0(B) = \begin{cases} F\text{: } \dim B = [F : \mathbf{Q}] \\ D/F\text{: totally indefinite division quaternion algebra over } F, \\ \qquad \dim(B) = 2[F : \mathbf{Q}] \end{cases}$$

If the following diagram commutes, we call B, $(\mathbf{Q}, F)$-abelian variety of GL_2-type.

$$\exists \varphi_\sigma : {}^\sigma B \longrightarrow B \ : \ D\text{-isogeny}$$

$$\begin{array}{ccc} {}^\sigma B & \xrightarrow{\varphi_\sigma} & B \\ {\scriptstyle {}^\sigma a}\downarrow & & \downarrow{\scriptstyle a} \\ {}^\sigma B & \xrightarrow[\varphi_\sigma]{} & B \end{array}$$

(2) $A/\overline{\mathbf{Q}}$: CM
$A \underset{/\overline{\mathbf{Q}}}{\sim} E^{\dim(A)}$, E : CM elliptic curve.

Conjecture 1 (Taniyama-Shimura, Ribet-Serre)
$A/\mathbf{Q}$: $\mathbf{Q}$-simple abelian variety of GL_2*-type.*
$\Longrightarrow$ *A is isogenous over $\mathbf{Q}$ to some $\mathbf{Q}$-simple factor of $J_1(N)$.*

Remark 2 *It is known that Conj. 1 is true for the following cases.*

(1) *A is a CM-abelian variety.*
(2) *A is an elliptic curve and* $27 \nmid \mathrm{cond}(A)$, (Wiles, Taylor, Diamond, Fontaine).

2 Moduli of GL_2-type

We fix a totally indefinite quaternion division algebra D/F.

$$d := \begin{cases} 1 & \text{if } D = F \\ \mathrm{disc}(D/F) & \text{if } D \neq F. \end{cases}$$

We consider the following three families of isogeny classes of abelian varieties.

(A) $\{(A,\iota) \mid A/\mathbf{Q} : \mathbf{Q}\text{-}simple\ abelian\ variety\ of\ \mathrm{GL}_2\text{-}type,\ \iota : F \hookrightarrow \mathrm{End}^0(A) = \mathrm{Mat}(D)\} / \underset{(A)}{\sim}$
(B) $\{(B,\iota) \mid B/\overline{\mathbf{Q}} : (\mathbf{Q},F)\text{-}abelian\ variety\ of\ \mathrm{GL}_2\text{-}type,\ \iota : F \hookrightarrow \mathrm{End}^0(B) = D\} / D\text{-isog}.$
(C) $\coprod_{\substack{\mathfrak{m} \subset \mathcal{O}_F \\ \text{square free} \\ (\mathfrak{m},d)=1}} (M(d,\mathfrak{m})/W)(\mathbf{Q})^0 / \underset{(C)}{\sim}.$

Notation

(A) $A_1 \underset{(A)}{\sim} A_2 \Longleftrightarrow \exists \chi$: *Dirichlet character,* $A_2 \underset{/\mathbf{Q}}{\sim} A_{1,\chi}$
with F-structure,($A_{1,\chi}$: twist of A_1 by χ).
(C) $M(d,\mathfrak{m})_{/\mathbf{Q}}$: *coarse moduli space of* $(B,\iota,V,\mathcal{E})$,
$\iota : \mathcal{O}_D \hookrightarrow \mathrm{End}(B)$,
$V \subset B[\mathfrak{m}]$, *$\mathcal{O}_D$-module,* $V \cong \mathcal{O}_D/\mathfrak{m}\mathcal{O}_D$,
$\mathrm{Trd}_{D/F}(a) = \mathrm{Tr}(a)$ *on* $\mathrm{Lie}B$ *as $\mathcal{O}_F$-modules* $(a \in \mathcal{O}_D)$,
$\mathcal{E}$: *canonical polarization.*

$W = W(d,\mathfrak{m}) \subset \mathrm{Aut}_{\mathbf{Q}} M(d,\mathfrak{m})$;
$\mathfrak{a},\mathfrak{p}$: *prime of* $\mathcal{O}_F$, $\mathfrak{P}$: *prime of* $\mathcal{O}_D$.

The following list is generators of W.

$$\begin{cases} W[\mathfrak{a}] : (B, *) \longmapsto (B/B[\mathfrak{a}], *'),\ (\mathfrak{a}, d\mathfrak{m}) = 1, (\mathfrak{a} \sim \mathfrak{a}' in\ C^{+}(F) \Longleftrightarrow W[\mathfrak{a}] = W[\mathfrak{a}']) \\ W[\mathfrak{p}] : (B, *) \longmapsto (B/B[\mathfrak{P}], *'),\ \mathfrak{p}|d,\ \mathfrak{P} \supset \mathfrak{p}, \\ W[\mathfrak{p}] : (B, V, *) \longmapsto (B/V[\mathfrak{p}], (V + B[\mathfrak{p}])/V[\mathfrak{p}], *'),\ \mathfrak{p}|d. \end{cases}$$

$(class\ of\ (B, *)) = x \in (M(d, \mathfrak{m})/W)(\mathbf{Q})^0$,
B : *simple abelian variety,*
$\forall \mathfrak{p} \mid \mathfrak{m},\ \exists \sigma \in \mathrm{G}_{\mathbf{Q}},\ \exists \gamma_a \in W\ s.t.\ \mathfrak{p}\ |deg\gamma_\sigma,\ {}^{\sigma}(B, *) \cong \gamma_a(B, *)$.
$\underset{(C)}{\sim};\ x \underset{(C)}{\sim} y \Longleftrightarrow \exists \mathfrak{n}(\neq 0) : ideal\ of\ \mathcal{O}_F,\ (\mathfrak{n}, d\mathfrak{m}) = 1,$
$\exists z \in (M(d, \mathfrak{m}\mathfrak{n})/W)(\mathbf{Q})$,
$\exists y = \pi w_{\mathfrak{n}}(z)$.
Here we consider W as a subgroup of $\mathrm{Aut}M(d, \mathfrak{m}\mathfrak{n})$.
$w_{\mathfrak{n}} : (B, V + V', *) \longmapsto (B/V', (V + B[\mathfrak{n}])/V', *'),\ V' \cong \mathcal{O}_D/\mathfrak{n}\mathcal{O}_D$.
$\pi : M(d, \mathfrak{m}\mathfrak{n}) \longrightarrow M(d, \mathfrak{m})$,
$(B, V + V', *) \longmapsto (B, V, *)$

Theorem 1 (A) $\overset{1:1}{\longleftrightarrow}$ (B) $\overset{1:1}{\longleftrightarrow}$ (C).

Remark 3
(A)$\overset{1:1}{\longleftrightarrow}$ (B) *is Ribet-Pyle's descent.*
(B)$\overset{1:1}{\longleftrightarrow}$(C) *is a generalization of the theory of Elkies's local tree.*

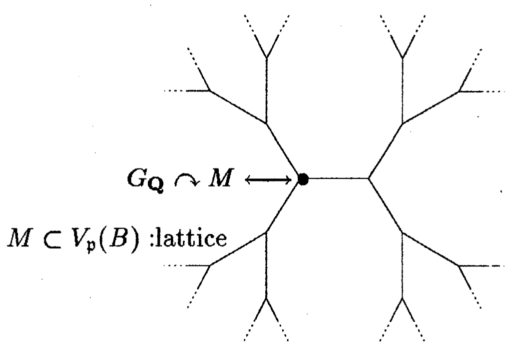
2.1 Trees

Let G be the tree attached to lattices of $V_{\mathfrak{p}}(B)$. More precisely, let $\varphi_\sigma : {}^{\sigma}B \longrightarrow B$ be a D-isogeny, then we have the following commutative diagram and i. (We note that $V_{\mathfrak{p}}(B)$ corresponds to $B[\mathfrak{p}^\infty]$.)

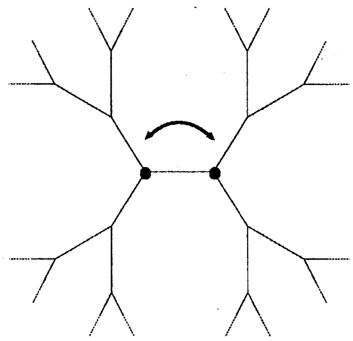
$$\begin{tikzcd} B[\mathfrak{p}^\infty] \arrow[r] \arrow[dr] & {}^{\sigma}B[\mathfrak{p}^\infty]. \\ & B[\mathfrak{p}^\infty]/B[\mathfrak{p}^i] \cong B[\mathfrak{p}^\infty] \arrow[u, "\mathcal{O}_D\text{-cyclic}"'] \end{tikzcd}$$

Hence, this $\mathcal{O}_D$-cyclic map gives the distance between $V_{\mathfrak{p}}(B)$ and ${}^{\sigma}V_{\mathfrak{p}}(B)$.
Because $G_{\mathbf{Q}}$ acts on the lattices, $G_{\mathbf{Q}}$ acts on G.

(1) $\mathfrak{p} \nmid d$,



(a) $G_{\mathbf{Q}} \curvearrowright G$ is fixed point free (in this case $\mathfrak{p} \mid \mathfrak{m}$),



• : the attached object is minimal.

(b) $G_{\mathbf{Q}} \curvearrowright G$ has fixed points (in this case $\mathfrak{p} \nmid \mathfrak{m}$),

• :fixed point, and the attached object is minimal.

(2) $\mathfrak{p} \mid d$,

$B[\mathfrak{p}^\infty]$ has a maximal $\mathcal{O}_D$-submodule $V = B[\mathfrak{P}] \subset B[\mathfrak{p}]$, $\mathfrak{P} \mid \mathfrak{p}$, $\mathfrak{P}^2 = \mathfrak{p}\mathcal{O}_D$.

$B[\mathfrak{p}^\infty] \longrightarrow B[\mathfrak{p}^\infty]/B[\mathfrak{P}] \longrightarrow (B[\mathfrak{p}^\infty]/B[\mathfrak{P}])/(B[\mathfrak{p}^\infty]/B[\mathfrak{P}])[\mathfrak{P}] = [B[\mathfrak{p}^\infty]/[B[\mathfrak{p}] = [B[\mathfrak{p}^\infty]$.

Hence, in this case G has exactly two vertices.

(a) $G_{\mathbf{Q}} \curvearrowright G$ is fixed point free (in this case $\mathfrak{p} \mid \delta$),

$\bullet$: the attached object is minimal.

(b) $G_{\mathbf{Q}} \curvearrowright G$ has fixed points (in this case $\mathfrak{p} \nmid \delta$),

$\bullet$:fixed point, and the attached object is minimal.

$(B) \longrightarrow (C)$
we choose locally minimal model for each $\mathfrak{p}$.
Global condition is $(B, *) \longmapsto w[\mathfrak{a}](B, *)$.
If there are more than two minimal points, we identify these by $(B, *) \longmapsto w_{\mathfrak{p}}(B, *)$ and $\widetilde{(C)}$.

$$(C)' : \ (M(d, \mathfrak{m})/W)(\mathbf{Q})^0 = (\coprod_{W' < W} (M(d, \mathfrak{m})/W')(\mathbf{Q})^{00})/W,$$

Here $x \in (M(d, \mathfrak{m})/W)(\mathbf{Q})^{00} \Longleftrightarrow G_{\mathbf{Q}}$ *acts on* $M(d, \mathfrak{m})_x$ *the fiber of* x *transitively.*

2.2 Invariant δ

(A) $\longleftrightarrow$ (B) $\longleftrightarrow$ (C)$'$.
$A/\mathbf{Q} \longleftrightarrow B/\overline{\mathbf{Q}} \longleftrightarrow x/\mathbf{Q}$.

$A/\mathbf{Q}$: Neben character ε of order 2-power.

Definition

(A) $S_A{}' := \{\mathfrak{p} \subset \mathcal{O}_F \mid \mathfrak{p} \mid D(K/F), \ \mathfrak{p} \nmid 2\}$.
$\delta_A{}' := \prod_{\mathfrak{p} \in S_A{}'} \mathfrak{p}$
(B) $S_B := \left\{\mathfrak{p} \subset \mathcal{O}_F \mid {}^{\exists\sigma \in G_{\mathbf{Q}},\ \exists D\text{-isogeny } {}^{\sigma}\psi : B \longrightarrow {}^{\sigma}B,}_{\mathfrak{p}\text{-part of } \ker({}^{\sigma}\psi) = \text{cyclic and odd degree}}\right\}$.
$\delta_B := \prod_{\mathfrak{p} \in S_A} \mathfrak{p}$.
$\delta_B{}' :=$ *2-primary part of* δ_B.
(C)$'$ $S_{C'} := \{\mathfrak{p} \subset \mathcal{O}_F \mid \exists \gamma \in W', \ \gamma = w[\mathfrak{a}] w_{\mathfrak{b}} w_{\mathfrak{c}}, \ (\mathfrak{a}, d\mathfrak{m}) = 1, \ \mathfrak{b} \mid d, \ \mathfrak{c} \mid \mathfrak{m}, \ \mathfrak{p} \mid \mathfrak{b}\mathfrak{c}\}$.
$\delta_{C'} := \prod_{\mathfrak{p} \in S_{C'}} \mathfrak{p}$.
$\delta_{C'}{}' :=$ *2-primary part of* $\delta_{C'}$.

Theorem 2
$\delta_B = \delta_{C'}$ (*We denote this value* δ).
$\delta_A{}' = \delta_B{}' = \delta_{C'}{}'$.

2.3 Invariant e_s and e

Definition
Let $\mathfrak{p} \mid d\mathfrak{m}$ *and* $\mathfrak{p} \mid p$. *For* $x \in (M(d,\mathfrak{m})/W')(\mathbf{Q})^{00}$,

$$e_s = \begin{cases} 2 & \text{if } W' \cap \mathrm{Aut}(B,*)/\overline{\mathbf{F}}_p \neq (1) \\ 1 & \text{if } W' \cap \mathrm{Aut}(B,*)/\overline{\mathbf{F}}_p = (1). \end{cases}$$

$e = \sharp\mathrm{Aut}(B,*)/\overline{\mathbf{F}}_p$.

3 Modularity

(class of $x = (B,*)) \in (M(d,\mathfrak{m})/W')(\mathbf{Q})^{00}$.
We assume that $\mathfrak{p} \mid d\mathfrak{m}$, $\mathfrak{p} \mid p \neq 2$ satisfy the following conditions;

$$(C1)\begin{cases} \mathfrak{p} \mid \mathfrak{m} & (B[\mathfrak{p}]/\overline{\mathbf{F}}_p)^{\text{ét}} \neq (0). \\ \mathfrak{p} \mid d & \mathrm{rank}_{\mathcal{O}_F/\mathfrak{p}\otimes\overline{\mathbf{F}}_p}\mathrm{Tan}_{/\overline{\mathbf{F}}_p}B[\mathfrak{P}] = 1,\ (\mathfrak{P} \subset \mathcal{O}_D,\ \mathfrak{P} \mid \mathfrak{p}). \end{cases}$$

$$(C2)\begin{cases} \mathfrak{p} \mid \mathfrak{m} & \begin{cases} p-1 \mid e & \text{if } e_s = 1,\ 4 \nmid e. \\ p-1 \mid \frac{1}{2}e_s e & \text{otherwise.} \end{cases} \\ \\ \mathfrak{p} \mid d & \end{cases}$$

(1) $B[p]/\overline{\mathbf{F}}_p)^{\text{ét}} = (0)$,

$$\begin{cases} \frac{1}{2}e \leq p-1,\ e \neq p+1 & \text{if } e_s = 1,\ 4 \nmid e. \\ \frac{1}{2}2^t e_s e \leq p-1 \text{ or } \frac{1}{2}2^t e_s e = 2(p-1) & \text{otherwise.} \end{cases}$$

Here $t := v_2(p-1)$.

(2) $B[p]/\overline{\mathbf{F}}_p)^{\text{ét}} \neq (0)$,

$$\begin{cases} \frac{1}{2}e \leq p-1,\ e \neq p+1 & \text{if } e_s = 1,\ 4 \nmid e. \\ \frac{1}{2}e_s e \leq p-1 & \text{otherwise.} \end{cases}$$

Proposition 1 *If* x *and* $\mathfrak{p} \mid d$ *satisfy* $(C1) \Longrightarrow \mathfrak{p} \mid \delta$.

Proposition 2 *We assume the condition* $(C1)$ *for* x *and* $\mathfrak{p}$. *We set* $e = p^r e'$, $p \nmid e'$. *Then*

(i) $\mathfrak{p} \mid \mathfrak{m}$,

$$\begin{cases} \mathrm{N}(\mathfrak{p}) \equiv 1 \pmod{e'} \\ \zeta_{p^r} \in F_{\mathfrak{p}}. \end{cases}$$

(ii) $\mathfrak{p} \mid d$,

$$\begin{cases} \mathrm{N}(\mathfrak{p}) \equiv -1 \pmod{e'} \\ v_{\mathfrak{p}}(p) \geq \varphi(p^r). \end{cases}$$

(iii) $B/\overline{\mathbf{F}}_p \sim E^{\dim B}$, E : *supersingular elliptic curve,*

Theorem 3
$x \in (M(d,\mathfrak{m})/W')(\mathbf{Q})^{00}$, $\exists \mathfrak{p} \mid d\mathfrak{m}$, $\mathfrak{p} \mid p \neq 2$ *such that satisfies* $(C1)$ *and* $(C2)$, *then the object of* x *is modular.*

Remark 4 *Under the above condition, we can show that*

$$\text{The Galois Image on } A[\mathfrak{p}] \subset \begin{cases} \text{Normalizer of Split Cartan subgroup} & \text{if } \mathfrak{p} \mid \mathfrak{m} \\ \text{Normalizer of Cartan subgroup} & \text{if } \mathfrak{p} \mid d. \end{cases}$$

Remark 5 *We omit the case* $\mathfrak{p} \mid \mathfrak{m}$ *and* $\mathfrak{p} \mid 3$. *But we can show the modularity with additional conditions. That is Theorem* 4.

Notation
$W'(\mathfrak{p}) < W'$: index 2, $\gamma \in W'(\mathfrak{p})$, $(\deg\gamma, \mathfrak{p}) = 1$, $\gamma = w[\mathfrak{a}]w_{\mathfrak{b}}w_{\mathfrak{c}}$,
$\deg\gamma = \mathfrak{abc}$, $(\mathfrak{a}, d\mathfrak{m}) = 1$, $\mathfrak{b} \mid d$, $\mathfrak{c} \mid \mathfrak{m}$.

Theorem 4
$x \in (M(d,\mathfrak{m})/W')(\mathbf{Q})^{00}$, $\exists \mathfrak{p} \mid d\mathfrak{m}$, $\mathfrak{p} \mid p = 3$ *such that*

(C0) $(M(d,\mathfrak{m})/W'(\mathfrak{p}))_x(\mathbf{R}) = \phi$.
(C1)$'$ $(B[\mathfrak{p}]/\overline{\mathbf{F}}_3)^{\acute{e}t} \neq (0)$.
(C2)$'$ $e_s = 1$, $e = 2$ (or 2×3-*power*).

$\Longrightarrow$ *the object of* x *is modular.*

Remark 6 *Moreover, we have a criterion of modularity for the case of the action of* $\mathrm{G}_{\mathbf{Q}}$ *on the tree has more than two fixed points by using the result of Skinner-Wiles.*

4 Q-curves

Let $D = F = \mathbf{Q}$, $\dim B = 1$.
We set $\mathfrak{m} = N$ (square free), $M(1, N) = X_0(N)$.

Theorem 5 *If $x \in (X_0(N)/W')(\mathbf{Q})^{00}$ satisfies the following conditions, then the object of x is modular.*

(A) *In the case of $p \geq 5$,*

$(C1)$ $x/\overline{\mathbf{F}}_p \neq$ supersingular point.

$(C2)$

e_s	e	p
1	2	$\geq 5\ (\neq 3)$
1	4	$\geq 5\ (\neq 3)$
1	6	$\geq 13\ (\neq 7)$
2	2	$\geq 5\ (\neq 3)$
2	4	$\geq 13\ (\neq 5)$
2	6	$\geq 13\ (\neq 7)$

(B) *In the case of $p = 3$,*

$(C0)$ $(X_0(N)/W'(3))_x(\mathbf{R}) = \phi$.

$(C2)'$ $e_s = 1,\ e = 2$.

Example 1

(4-1) $N = p \geq 5, \neq 7$;
If $x \in (X_0(p)/w_p)(\mathbf{Q})$ is non-cuspidal point and if $x \pmod p$ is non-supersingular point, then x : modular.
If $p = 7$ and $e \neq 6$, then modular.

(4-2) $N = 35, 39$ ($\Longrightarrow X_0(N)/w_N = \mathbf{P}^1$);
$x \in (X_0(N)/w_N)(\mathbf{Q})$: non-cuspidal point $\Longrightarrow x$: modular.
If $N = 65$, then $X_0(N)/w_N$ is an elliptic curve with positive rank.
$x \in (X_0(N)/w_N)(\mathbf{Q})$: non-cuspidal point $\Longrightarrow x$: modular.

(4-3) $N = p = 3$;
By using the Fricke's explicit defining equation of $X_0(3)$:

$$\begin{cases} j := j(\tau) = 27(\tau+1)(9\tau+1)^3/\tau, \\ w_3(\tau) = 1/\tau. \end{cases}$$

we obtain the conditions in Theorem. 4 explicitly as follows (v_3 : the valuation at 3).
$(C0)$ implies τ is a root of $X^2 - aX + 1 = 0$, $a \in \mathbf{Q}$ and $|a| < 2$.
$(C1)$ implies $v_3(j) \leq 0$. It implies $v_3(a) \geq 1$ or $v_3(a) \leq 3$.

If τ is a root of $X^2 - aX + 1 = 0$ where a satisfies above conditions, then $j(\tau)$ gives modular $\mathbf{Q}$-curve.

5 QM-abelian surfaces

Let $D/\mathbf{Q}$ be a indefinite quaternion division algebra. (i.e. $D \neq F$, $F = \mathbf{Q}$, $\dim B = 2$)
$M(d,m)/\mathbf{Q}$: Shimura curve.

Theorem 6 *For $p \mid dm$, $p \neq 2$,*
If $x \in (M(d,m)/W')(\mathbf{Q})^{00}$ satisfies the following conditions, then the object of x is modular.

(1) $p \mid m$; *Conditions $(C1)$ and $(C2)$ are same as of $\mathbf{Q}$-curves.*

(2) $p \mid d$; *For $y \in M(d,m)_x$,*

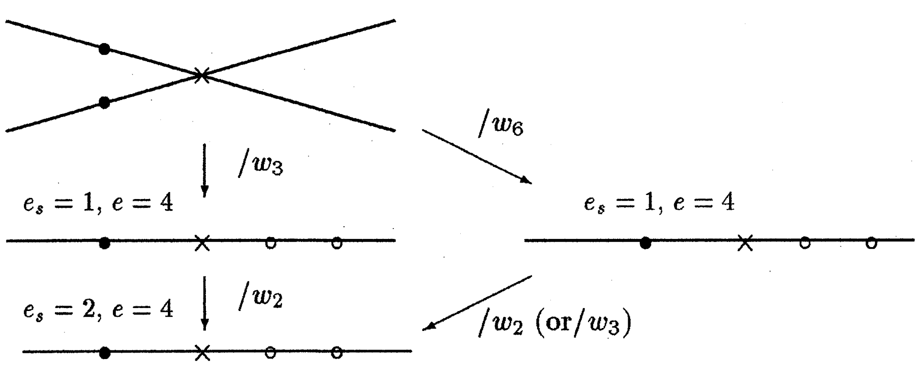
$(C1)$ $y/\overline{\mathbf{F}}_p \neq$ double point.

$(C2)$

e_s	e	p
1	2	≥ 3
1	4	≥ 3
1	6	≥ 11 $(\neq 5)$
2	2	≥ 3
2	4	≥ 11 $(\neq 3, 7)$
2	6	$\neq 1 + 2$-*power*$, 1 + 5 \times 2$-*power*.

Example 2

(5-1) *The following diagram is the covering map of $M(6,1)$ reduction mod 3 and its quotient curves by its involutions. ($M(6,1)/w_a = \mathbf{P}^1$, $a \neq 1$)*



$\times$: $e = 6$, $\bullet$: $e = 4$, $\circ$: $\mathbf{F}_3$-rational points ($e_s = 1$, $e = 2$)

(5-2) *If d is contained in the following list, then $M(d,1)/w_d = \mathbf{P}^1$ and $\forall x \in (M(d,1)/w_d)(\mathbf{Q})$ is modular.*

$$d = 14, 21, 33, 34, 35, 39, 46, 51, 55, 62, 69, 87, 94, 95, 111, 119, 159.$$

If d is contained in the following list, then $M(d,1)/w_d$ is an elliptic curves with positive rank and $\forall x \in (M(d,1)/w_d)(\mathbf{Q})$ is modular.

$$d = 57, 65, 77, 129, 143.$$

Remark 7 *Even for $m \neq 1$, We have obtained some examples.*